§3 链式求导法则

一、多元函数求导的链式法则

定理 设二元函数 $u = f(y_1, y_2)$ 可微，

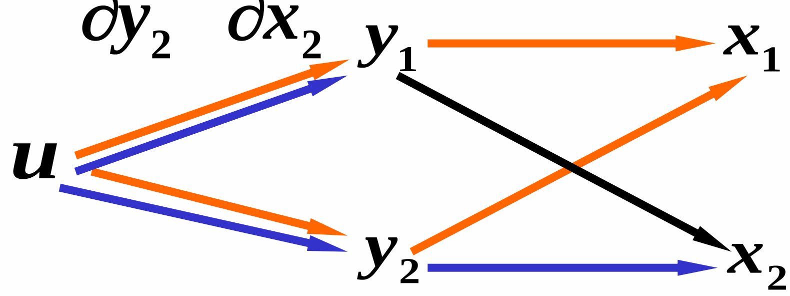
二个二元函数 $\begin{cases} y_1 = g_1(x_1, x_2) \\ y_2 = g_2(x_1, x_2) \end{cases}$ 可微，

则 u 作为 (x_1, x_2) 的函数是可微的，

且 $$\frac{\partial u}{\partial x_1} = \frac{\partial u}{\partial y_1}\cdot\frac{\partial y_1}{\partial x_1} + \frac{\partial u}{\partial y_2}\cdot\frac{\partial y_2}{\partial x_1}$$

$$\frac{\partial u}{\partial x_2} = \frac{\partial u}{\partial y_1}\cdot\frac{\partial y_1}{\partial x_2} + \frac{\partial u}{\partial y_2}\cdot\frac{\partial y_2}{\partial x_2}$$

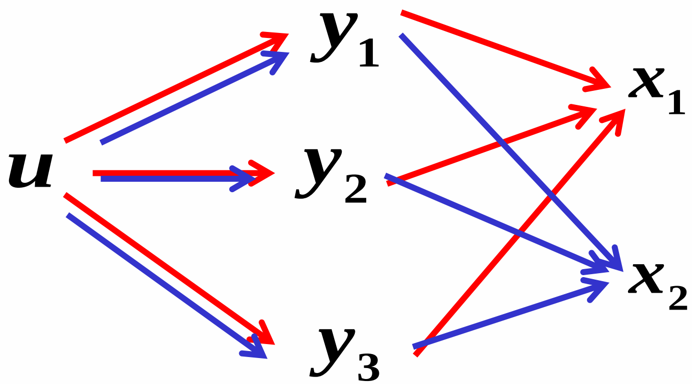
链式法则如图示



对三元函数 $u = f(y_1, y_2, y_3)$ 可微,

三个二元函数 $\begin{cases} y_1 = g_1(x_1, x_2) \\ y_2 = g_2(x_1, x_2) \\ y_3 = g_3(x_1, x_2) \end{cases}$

$$\frac{\partial u}{\partial x_i} = \frac{\partial u}{\partial y_1} \cdot \frac{\partial y_1}{\partial x_i} + \frac{\partial u}{\partial y_2} \cdot \frac{\partial y_2}{\partial x_i} + \frac{\partial u}{\partial y_3} \cdot \frac{\partial y_3}{\partial x_i} \quad i = 1, 2$$



对 m 元函数 $u=f(y_1,\cdots,y_m)$ 可微，

n 个 m 元函数 $\begin{cases} y_1=g_1(x_1,\cdots,x_n) \\ \vdots \\ y_m=g_m(x_1,\cdots,x_n) \end{cases}$ 可微，

则 u 作为 $(x_1,\cdots,x_n)$ 的函数是可微的，

且 $$\frac{\partial u}{\partial x_i}=\sum_{j=1}^{m}\frac{\partial u}{\partial y_j}\frac{\partial y_j}{\partial x_i} \quad i=1,\cdots,n$$

实质:

因变量 u 关于自变量 x_i 的偏导数，等于u 关于各中间变量的偏导数与该中间变量关于 x_i 的偏导数乘积之和。

例1、设 $z=f(x,y)=e^{x}\sin y$，$x=st$，$y=\dfrac{t}{s}$.

求 $\dfrac{\partial z}{\partial s}$，$\dfrac{\partial z}{\partial t}$.

解：$$\frac{\partial z}{\partial s}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s}=e^{x}\sin y\cdot t+e^{x}\cos y\cdot\left(-\frac{t}{s^{2}}\right)$$

$$=te^{st}\left(\sin\frac{t}{s}-\frac{1}{s^{2}}\cos\frac{t}{s}\right)$$

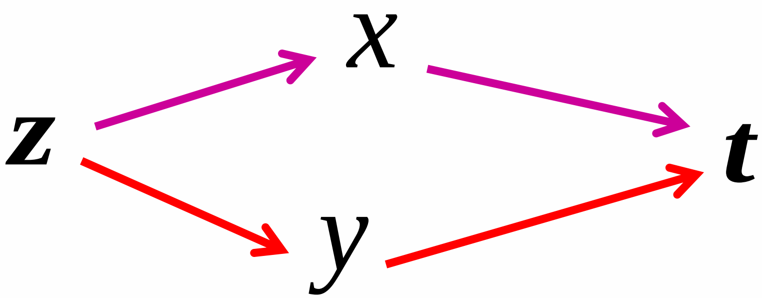
$$\frac{\partial z}{\partial t}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t}=e^{x}\sin y\cdot s+e^{x}\cos y\cdot\frac{1}{s}$$

$$=e^{st}\left(s\sin\frac{t}{s}+\frac{1}{s}\cos\frac{t}{s}\right)$$

在复合函数中若只有一个自变量，

即 $z=f(x,y)\quad x=\varphi(t)\quad y=\psi(t)$

$$\Rightarrow \frac{dz}{dt}=\frac{\partial z}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$ 称为全导数。



例2、设 $z=\tan(3t+2x^2-y)$， $x=\frac{1}{t}$， $y=\sqrt{t}$，求 $\frac{dz}{dt}$.

解： $$\frac{dz}{dt}=\frac{\partial z}{\partial t}\cdot\frac{dt}{dt}+\frac{\partial z}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$

$$=\sec^2(3t+2x^2-y)\cdot 3$$

$$+\sec^2(3t+2x^2-y)\cdot 4x\cdot(-\frac{1}{t^2})$$

$$+\sec^2(3t+2x^2-y)\cdot(-1)\cdot\frac{1}{2}t^{-\frac{1}{2}}$$

$$=(3-\frac{4}{t^3}-\frac{1}{2\sqrt{t}})\sec^2(3t+\frac{2}{t^2}-\sqrt{t})$$

例3、设 $u=e^{x^2+y^2+z^2}$，$z=x^2\sin y$，求 $\frac{\partial u}{\partial x}$，$\left.\frac{\partial u}{\partial y}\right|_{(1,\frac{\pi}{2})}$.

解： 记 $u=f(x,y,z)=e^{x^2+y^2+z^2}$

$$\frac{\partial u}{\partial x}=\frac{\partial f}{\partial x}\cdot\frac{dx}{dx}+\frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial x}+\frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial x}=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial x}$$

$$=2xe^{x^2+y^2+z^2}+2ze^{x^2+y^2+z^2}\cdot 2x\sin y$$

$$=2x(1+2x^2\sin^2 y)e^{x^2+y^2+x^4\sin^2 y}$$

$$\frac{\partial u}{\partial y}=\frac{\partial f}{\partial y}+\frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial y}$$

$$=2ye^{x^2+y^2+z^2}+2ze^{x^2+y^2+z^2}\cdot x^2\cos y$$

$$=2(y+x^4\sin y\cos y)e^{x^2+y^2+x^4\sin^2 y}$$

$$\therefore \left.\frac{\partial u}{\partial y}\right|_{(1,\frac{\pi}{2})}=\pi e^{2+\frac{\pi^2}{4}}$$

特殊地 $z = f(u, x, y)$ 其中 $u = \phi(x, y)$

即 $z = f[\phi(x, y), x, y]$,

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial f}{\partial x}, \quad \frac{\partial z}{\partial y} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial f}{\partial y}.$$

区别类似

两者的区别

把复合函数 $z = f[\phi(x, y), x, y]$ 中的 y 看作不变而对 x 的偏导数

把 $z = f(u, x, y)$ 中的 u 及 y 看作不变而对 x 的偏导数

例4、设 $z=xy+xF(u)$，$u=\dfrac{y}{x}$， 求 $\dfrac{\partial z}{\partial x}$，$\dfrac{\partial z}{\partial y}$.

解： $z=f(x,y,u)=xy+xF(u)$

$$\frac{\partial z}{\partial x}=\frac{\partial f}{\partial x}\cdot\frac{dx}{dx}+\frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial x}+\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x}=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x}$$

$$=y+F(u)+xF'(u)\left(-\frac{y}{x^2}\right)$$

$$=y+F\left(\frac{y}{x}\right)-\frac{y}{x}F'\left(\frac{y}{x}\right)$$

$$\frac{\partial z}{\partial y}=\frac{\partial f}{\partial y}+\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y}=x+xF'(u)\left(\frac{1}{x}\right)$$

$$=x+F'\left(\frac{y}{x}\right)$$

例5、设 $z=x^n f\left(\dfrac{y}{x^2}\right)$，其中 f 为任意可微函数，

求证 $x\dfrac{\partial z}{\partial x}+2y\dfrac{\partial z}{\partial y}=nz$.

证：

$$\frac{\partial z}{\partial x}=nx^{n-1}f\left(\frac{y}{x^2}\right)+x^n f'\left(\frac{y}{x^2}\right)\cdot\left(-\frac{2y}{x^3}\right)$$

$$=nx^{n-1}f\left(\frac{y}{x^2}\right)-2x^{n-3}yf'\left(\frac{y}{x^2}\right)$$

$$\frac{\partial z}{\partial y}=x^n f'\left(\frac{y}{x^2}\right)\cdot\left(\frac{1}{x^2}\right)=x^{n-2}f'\left(\frac{y}{x^2}\right)$$

$$\therefore x\frac{\partial z}{\partial x}+2y\frac{\partial z}{\partial y}$$

$$=x\left[nx^{n-1}f\left(\frac{y}{x^2}\right)-2x^{n-3}yf'\left(\frac{y}{x^2}\right)\right]+2yx^{n-2}f'\left(\frac{y}{x^2}\right)$$

$$=nz.$$

例6、设 $w=f(x+y+z,xyz)$，

f 具有二阶连续偏导数，求：$\dfrac{\partial w}{\partial x}$，$\dfrac{\partial^2 w}{\partial x\partial z}$．

解：令 $u=x+y+z$，$v=xyz$，

记 $f_1'=\dfrac{\partial f(u,v)}{\partial u}$，$f_{12}''=\dfrac{\partial^2 f(u,v)}{\partial u\partial v}$，

同理有 f_2'，f_{11}''，f_{22}''，

$$\frac{\partial w}{\partial x}=\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x}=f_1'+yz\,f_2'$$

$$\frac{\partial^2 w}{\partial x\partial z}=\frac{\partial}{\partial z}(f_1'+yz\,f_2')=\frac{\partial f_1'}{\partial z}+yf_2'+yz\frac{\partial f_2'}{\partial z}$$

$$\frac{\partial^2 w}{\partial x \partial z} = \frac{\partial f_1'}{\partial z} + y f_2' + yz \frac{\partial f_2'}{\partial z}$$

$$\frac{\partial f_1'}{\partial z} = \frac{\partial f_1'}{\partial u} \cdot \frac{\partial u}{\partial z} + \frac{\partial f_1'}{\partial v} \cdot \frac{\partial v}{\partial z} = f_{11}'' + xy\, f_{12}''$$

$$\frac{\partial f_2'}{\partial z} = \frac{\partial f_2'}{\partial u} \cdot \frac{\partial u}{\partial z} + \frac{\partial f_2'}{\partial v} \cdot \frac{\partial v}{\partial z} = f_{21}'' + xy\, f_{22}''$$

$$\therefore \frac{\partial^2 w}{\partial x \partial z} = f_{11}'' + xy\, f_{12}'' + y\, f_2' + yz(f_{21}'' + xy\, f_{22}'')$$

$$= f_{11}'' + y(x+z)\, f_{12}'' + xy^2 z\, f_{22}'' + y\, f_2'$$

二、全微分形式不变性

设以 $y=(y_1,y_2)$ 为自变量的二元函数，

$u=f(y_1,y_2)$ 可微，

则其全微分 $du=\dfrac{\partial u}{\partial y_1}dy_1+\dfrac{\partial u}{\partial y_2}dy_2$

如果变量 $y_i\ (i=1,2)$ 又是变量 (x_1,x_2) 的可微函数，

则其全微分 $dy_1=\dfrac{\partial y_1}{\partial x_1}dx_1+\dfrac{\partial y_1}{\partial x_2}dx_2$

$$dy_2=\frac{\partial y_2}{\partial x_1}dx_1+\frac{\partial y_2}{\partial x_2}dx_2$$

$$du=\frac{\partial u}{\partial x_1}dx_1+\frac{\partial u}{\partial x_2}dx_2$$

$$=(\frac{\partial u}{\partial y_1}\frac{\partial y_1}{\partial x_1}+\frac{\partial u}{\partial y_2}\frac{\partial y_2}{\partial x_1})dx_1+(\frac{\partial u}{\partial y_1}\frac{\partial y_1}{\partial x_2}+\frac{\partial u}{\partial y_2}\frac{\partial y_2}{\partial x_2})dx_2$$

$$=\frac{\partial u}{\partial y_1}(\frac{\partial y_1}{\partial x_1}dx_1+\frac{\partial y_1}{\partial x_2}dx_2)+\frac{\partial u}{\partial y_2}(\frac{\partial y_2}{\partial x_1}dx_1+\frac{\partial y_2}{\partial x_2}dx_2)$$

$$=\frac{\partial u}{\partial y_1}dy_1+\frac{\partial u}{\partial y_2}dy_2$$

结论 无论函数 f 看作自变量 (x_1, x_2) 的函数，

还是看作中间变量 (y_1, y_2) 的函数，

其全微分的形式不变。

思考题

设 $z=f(u,v,x)$，$u=\phi(x)$，$v=\varphi(x)$，

则 $$\frac{dz}{dx}=\frac{\partial f}{\partial u}\frac{du}{dx}+\frac{\partial f}{\partial v}\frac{dv}{dx}+\frac{\partial f}{\partial x}$$

试问 $\frac{dz}{dx}$ 与 $\frac{\partial f}{dx}$ 是否相同？为什么？